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OPTIMIZATION APPROACH FOR SOLVING THE CONTINUATION PROBLEM IN FREQUENCY DOMAIN

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Abstract. The paper presents analytical expressions for solving the problem of continuation of the electromagnetic field in the frequency domain for a horizontally layered medium. The numerical solution algorithm uses layerwise recalculation of the required quantities, for which analytical expressions are presented in a form that allows calculations and layerwise recalculation without the accumulation of rounding errors. The solution to the continuation problem is obtained as a solution to the cost functional minimization problem. The strong convexity of the functional is proven, which implies the existence of a unique solution to the problem posed. Before solving the continuation problem, it is proposed to apply the Butterworth filter to smooth the radargrams and the Hilbert transform technique to clarify the coordinates of the discontinuity points of the medium.

Keywords: continuation problem, frequency domain, horizontally layered medium, Hilbert transform, global minimum.

AMS Subject Classification: 35A24, 35A25, 35B60, 35Q61, 35R30.

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1 Introduction

The paper presents analytical expressions for solving the problem of continuation of the electromagnetic field in the frequency domain for horizontally layered media. The numerical solution algorithm is based on layer-by-layer recalculation. The resulting analytical expressions allow recurrent calculations without accumulation of rounding errors. An optimization method was used to solve the continuation problem. The strong convexity of the residual functional is proven, which implies the existence of a unique solution to the problem posed.

This problem of continuation of the electromagnetic field to a certain depth is relevant when the electromagnetic properties of the first few layers are known, and below the electromagnetic properties of the layered medium are subject to determination. If the continuation problem is solved, then the direct and inverse problems can be formulated only in the subdomain where the electromagnetic parameters are unknown. Since the subdomain is smaller than the entire domain, this means that the time to solve the direct problem, and therefore the time to solve the inverse problem, may require significantly less time.

One of the first technologically advanced layer-by-layer recalculation algorithms for solving a boundary value problem for second-order differential equation for a horizontally layered medium

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was algorithm in the work (Tikhonov & Shakhsuvarov, 1956). However, it had some limitations: analytical expressions contained exponential functions whose exponents had positive real parts, which led to the accumulation of rounding errors during calculations and layer-by-layer recalculation. This drawback of the proposed layer-by-layer recalculation method was eliminated in the Dmitriev's works (see, for example, Dmitriev, 1968; Dmitriev & Fedorova, 1980). To construct expressions for calculations, he used the idea of Gelfand & Lokutsievsky, 1962, for solving a boundary value problem for second-order differential equation. It made possible to obtain expressions for calculations that are resistant to the accumulation of rounding errors. Further, this method of layer-by-layer recalculation was developed for solving systems of second-order differential equations: for a system of equations of the theory of elasticity (horizontally layered isotropic medium (Akkuratov & Dmitriev, 1979, 1984, Pavlov, 2002), isotropic medium with absorption (Fatianov & Mikhailenko, 1988; Fatianov, 1990), transversely isotopic medium with a symmetry axis coinciding with the Oz axis (Fatianov, 1989), a medium of any type of anisotropy (Karchevsky, 2005a; 2005b), for calculating the gradient of the residual functional for solving inverse problems to determine the velocity parameters of thinly-stratified layer (Karchevsky, 2003; Kurpinar & Karchevsky, 2005), for Maxwell's equations (horizontally layered medium of any type of anisotropy (Karchevsky, 2007), for the system of equations of convective heat and moisture transfer (Karchevsky & Rysbayuly, 2015), for the fourth-order differential equation of transverse vibrations of a piecewise homogeneous beam (Karchevsky, 2020a). Currently, the layer-by-layer recalculation method for solving second-order differential equations or systems of second-order differential equations for horizontally layered media are recognized as the most suitable for calculations (Somersalo et al, 1991). In this work, the layer-by-layer recalculation method is applied to finding expressions that are resistant to the accumulation of rounding errors, not for the numerical solution of the direct problem, but for the numerical solution of the continuation problem of the electromagnetic field in the frequency domain.

The layer-by-layer recalculation method makes it possible to obtain a numerical method for solving the direct problem that is not only resistant to rounding errors but also requires relatively little time for calculations. This is a very useful property, especially when solving inverse problems using the optimization method, since in this case solving the inverse problem is reduced to repeatedly solving the direct problem (see, for example, Gribov et al., 2019; Abgarian et al., 2017; Popov & Tikhonravov, 1997; Habashy & Mittra, 1987; Aida-Zade & Kuliev, 2011; Abdullaev & Aida-Zade, 2014; Durdiev, 2020; Bozorov, 2020; Karchevsky, 2000; 2004; 2009; Karchevsky & Fatianov, 2001; Duchkov & Karchevsky, 2013; Romanov & Karchevsky, 2018). The time for solving the direct and, therefore, the inverse problem can be greatly reduced if it is necessary to solve the inverse problem of determining the parameters of the medium, starting from a certain depth, since the upper part of the medium is known. This could be, for example, a road surface, an airfield runway, the upper part of an archaeological section, etc.

The numerical solution of field continuation problems is, as a rule, unstable, and therefore requires the use of regularization methods (see, for example, Tikhonov, 2009; Tikhonov & Arsenin, 1979; Lavrent'ev et al., 1969; 1980; Vasin & Ageev, 1993; Vasin & Eremin, 2005; Yagola et al., 2014). 2D and 3D continuation problems in the stationary case is mathematically formulated as the Cauchy problem for an elliptic equation, in the non-stationary case – as a problem for a parabolic or hyperbolic equation with data on a timelike boundary. For the numerical solution of the Cauchy problem for an elliptic equation, many methods for numerical solution have been proposed (see the review of methods in the work (Sibiryakov et al., 2023), for the numerical solution of equations with data on the timelike boundary, we know only two methods – the optimization method (Kabanikhin et al., 2013; Belonosov & Shishlenin, 2016) and the adjoint operator method (Karchevsky, 2013). As a rule, the proposed methods for solving the Cauchy problem (see (Sibiryakov et al, 2023)) were tested on simulated data, in the works (Karchevsky et al., 2016; Cheverda et al., 2016; 2017) the problem was solved using real data obtained during laboratory experiments. Also, the solution to a parabolic equation with data on a timelike boundary was obtained after laboratory measurements (Karchevsky 2018; 2020b). Some simple statements of continuation problems can be found in (Yaparova, 2012; 2013; Solodusha & Yaparova, 2015).

In our case, when the field continuation problem is solved in the frequency domain, it is necessary to obtain analytical expressions for solving the Cauchy problem for a second-order differential equation. This is a well-posed problem. This means that a correct method for solving this problem must be presented. Here we propose method based on layer-by-layer recalculation: in each layer, analytical expressions are obtained for solving the problem such that numerical calculations and recurrent recalculation do not accumulate rounding errors.

2 Mathematical statement of the problem

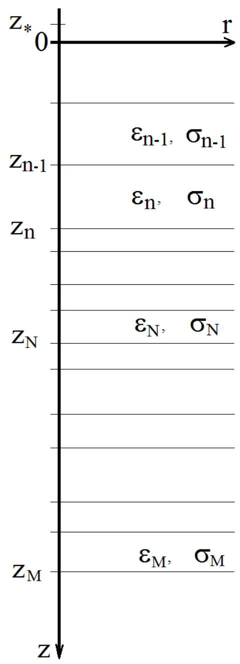

Figure 1: Medium model

Let the medium model be a horizontally-layered structure with interfaces z_n $(n = \overline{0,M})$, the n-th layer is the interval $[z_{n-1}, z_n]$, $h_n = z_n - z_{n-1}$ – thickness of the n-th layer, $(-\infty, z_0]$ is air, $[z_M, \infty)$ is underlying half-space. The propagation of the electromagnetic field is described by the Maxwell equations. Medium properties is determined by the permittivity ε, the conductivity σ and the magnetic permeability μ. The permittivity $\varepsilon = \epsilon_0\epsilon$ where $\epsilon_0 = 8.854\cdot 10^{-12}$(F/m) is the permittivity of vacuum and $\epsilon \geq 1$ is the relative permittivity of the medium. For most geophysical media $\mu = 4\pi\cdot 10^{-7}$(H/m). For air $\epsilon = 1$ and $\sigma = 0$. We will assume that ϵ and σ depend only on depth, i.e. $\epsilon = \epsilon(z)$ and $\sigma = \sigma(z)$ and are piecewise constant functions. The electromagnetic properties of the n-th layer are determined by the constants ϵ_n and σ_n.

Assume that the electromagnetic field is excited by the external current source

$$
\begin{aligned}
j(r,\phi,z,t) &= \begin{pmatrix} j_r \\ j_\phi \\ j_z \end{pmatrix} \equiv \begin{pmatrix} 0 \\ j_\phi \\ 0 \end{pmatrix}, \\
j_\phi(r,z,t) &= f(t)\delta(r-r_0)\delta(z-z_*), \qquad (1)
\end{aligned}
$$

where z_* is the source location, $z_* < 0$ (the source is in the air), z_* is small enough, and $r_0 > 0$ is the source parameter.

Since the medium is assumed to be isotropic and the source does not depend on the angle ϕ, Maxwell's equations can be written in cylindrical coordinates, and the components of the electromagnetic field not depend on the angle ϕ. Considering the source type (1), only three of the six components of the electromagnetic field are non-zero: E_ϕ, H_r and H_z (see, for example, Romanov & Kabanikhin, 1994; Romanov & Karchevsky, 2018). For the component E_ϕ the following differential equation can be obtained

$$
\varepsilon\frac{\partial^2 E_\phi}{\partial t^2} + \sigma\frac{\partial E_\phi}{\partial t} = \frac{1}{\mu}\left[\frac{\partial^2 E_\phi}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{1}{\partial r}(rE_\phi)\right)\right] - \frac{\partial j_\phi}{\partial t}. \qquad (2)
$$

At the initial moment of time there is no electromagnetic field. Receivers recording electrograms are located along the r axis. We apply the Laplace and the Hankel transforms to the function $E_\phi(r,z,t)$:

$$
w(\nu,z,p) = \int_0^\infty e^{-pt}\int_0^\infty rE_\phi(r,z,t)J_1(\nu r)drdt, \qquad (3)
$$

where $p = \chi - i2\pi \mathrm{f}$ is the Laplace transform parameter (χ is the attenuation parameter, f is the time frequency (Hz)), $J_1(r)$ is the first order Bessel function (see, for example, Gradshtein & Ryzhik, 1963; Janke et al., 1960), ν is the Hankel transform parameter.

Below we will use the following notations: $f(p)$ – the Laplace image for the function $f(t)$, $\kappa(z) = \sqrt{\nu^2 + p^2\mu\epsilon_0\epsilon(z) + p\mu\sigma(z)}$, where $\mathrm{Re}\{\kappa(z)\} > 0$, for n-th layer $\kappa_n = \sqrt{\nu^2 + p^2\mu\epsilon_0\epsilon_n + p\mu\sigma_n}$, for air $\kappa_0 = \sqrt{\nu^2 + p^2\mu\epsilon_0}$, and $g(\nu) = r_0 J_1(\nu r_0)$.

It is easy to see that for $z \in \{(-\infty, \infty)\backslash\{z_0\}...\backslash\{z_M\}\}$ the function $w(\nu, z, p)$ satisfies the following differential equation:

$$w_{zz} - \kappa^2(z)w = \mu p f(p) g(\nu)\delta(z - z_*). \tag{4}$$

The continuity of the tangential components of the Maxwell system when passing through the discontinuity points of the medium $z = z_n$ ($n = \overline{0, M}$) entails the satisfaction of the gluing conditions:

$$[w_z]_{z_n} = 0, \quad [w]_{z_n} = 0, \tag{5}$$

Additionally, it must be assumed that the damping conditions

$$w|_{z \to \pm\infty} = 0 \tag{6}$$

occurs.

Since $\delta(z - z_*)$ is present in the source (1), we can consider the differential equation as homogeneous, but there are additional gluing conditions at the point z_*:

$$[w_z]_{z_*} = \mu p f(p) g(\nu), \quad [w]_{z_*} = 0. \tag{7}$$

Taking into account the second damping condition (6), the gluing conditions at the point z_* (7) and at the point z_0 (5), the smallness of the value z_*, we will calculate the limit when $z_* \to 0$ and obtain the boundary condition at the point $z_0 = 0$. Finally, we have the following formulation of the direct problem:

$$w_{zz} - \kappa^2 w = 0, \tag{8}$$

$$(w_z - \kappa_0 w)|_{z=0} = \mu p f(p) g(\nu), \quad w|_{z \to \infty} = 0, \tag{9}$$

$$[w_z]_{z_n} = 0, \quad [w]_{z_n} = 0, \quad n = \overline{1, M}. \tag{10}$$

The inverse problem can be posed: Determine the electromagnetic parameters of the medium ϵ_n and σ_n, if additional information

$$w|_{z=0} = \psi(\nu, p), \tag{11}$$

is known about the solution of the direct problem (8)-(10) (here $\psi(\nu, p)$ is the Laplace image for $E_\phi|_{z=0} = \psi(r, t)$).

This inverse problem can be solved by minimizing the residual functional:

$$\Phi[\epsilon_n, \sigma_n] = \sum_{i,j} |w|_{z=0} - \psi(\nu_i, p_j)|^2, \tag{12}$$

(ν_i and f_j ($p_j = \chi - i2\pi\mathrm{f}_j$) belong to finite sets).

However, if ϵ_n and σ_n ($n = \overline{1, N}$) are known, then it is advisable to solve the continuation problem

$$w_{zz} - \kappa^2 w = 0, \tag{13}$$

$$(w_z - \kappa_0 w)|_{z=0} = \mu p f(p) g(\nu), \quad w|_{z=0} = \psi(\nu, p), \tag{14}$$

$$[w_z]_{z_n} = 0, \quad [w]_{z_n} = 0, \quad n = \overline{1, N-1}, \tag{15}$$

and determine the values

$$w_z|_{z=z_N} \quad \text{and} \quad w|_{z=z_N}. \tag{16}$$

After this, we can pose a direct problem that will be solved on an interval that is smaller than the interval on which the direct problem (8)-(10) is solved. For this statement of direct problem, we can pose the statement of the inverse problem and write the residual functional. Since the interval is smaller, minimizing the residual functional will take less time than minimizing the residual functional (12).

The purpose of this work to propose a stable algorithm for the numerical determination of the values (16).

3 Derivating main relations

We consider the new functions $a(z)$ and $b(z)$, which are connected with the function $w(\nu, z, p)$:

$$w_z(\nu, z, p) = a(z)w(\nu, z, p) + b(\nu, z, p). \tag{17}$$

It should be noted here that since the constants ν and p are included as parameters in the formulation of (8)-(10), (11) then the functions $a(z)$ and $b(z)$ depend on them, but for brevity this dependence is not indicated.

Substituting the relation (17) into (8)-(10), we obtain that the functions $a(z)$ and $b(z)$ satisfy the following problems:

$$a' + a^2 = \kappa^2, \quad a(0) = \kappa_0, \quad [a]_{z_n} = 0 \ \ (n = \overline{1, N-1}), \tag{18}$$

$$b' + ab = 0, \quad b(0) = \mu p f(p) g(\nu), \quad [b]_{z_n} = 0 \ \ (n = \overline{1, N-1}). \tag{19}$$

We denote $a^n = a(z_n)$, $b^n = b(z_n)$, $w_n = w(\nu, z_n, p)$ for brevity.

In each n-th layer, the solution to the Riccati differential equation (18) can be represented in the following form:

$$a(z) = \kappa_n \frac{(a^{n-1} + \kappa_n) + (a^{n-1} - \kappa_n)e^{-2\kappa_n(z - z_{n-1})}}{(a^{n-1} + \kappa_n) - (a^{n-1} - \kappa_n)e^{-2\kappa_n(z - z_{n-1})}}. \tag{20}$$

Using the gluing conditions (18) we obtain that a^n can be defined recursively:

$$a^0 = \kappa_0, \quad a^n = \kappa_n \frac{(a^{n-1} + \kappa_n) + (a^{n-1} - \kappa_n)e^{-2\kappa_n h_n}}{(a^{n-1} + \kappa_n) - (a^{n-1} - \kappa_n)e^{-2\kappa_n h_n}}, \quad n = \overline{1, N}. \tag{21}$$

In each n-th layer, the solution to the differential equation from (19) can be represented in the following form:

$$b(z) = b^{n-1} e^{-\int\limits_{z_{n-1}}^{z} a(x)dx} = b^{n-1} \frac{2\kappa_n e^{-\kappa_n(z - z_{n-1})}}{(a^{n-1} + \kappa_n) - (a^{n-1} - \kappa_n)e^{-2\kappa_n(z - z_{n-1})}}. \tag{22}$$

Using the gluing conditions (19) we obtain that b^n can be determined recurrently:

$$b^0 = \mu p f(p) g(\nu), \quad b^n = b^{n-1} \frac{2\kappa_n e^{-\kappa_n h_n}}{(a^{n-1} + \kappa_n) - (a^{n-1} - \kappa_n)e^{-2\kappa_n h_n}}, \quad n = \overline{1, N}. \tag{23}$$

Easy to see, the values a^n and b^n are calculated “top to bottom”.

The expressions (20)-(23) have only exponential functions whose real parts of the exponents are negative. This means that recurrent recalculation will be done without the accumulation of rounding errors.

In order to obtain expressions that are resistant to rounding errors, in each n-th layer the solution to the differential equation (17) can be solved "bottom to top":

$$\begin{aligned} w(\nu,z,p) &= w_n\, e^{\int\limits_{z_n}^{z} a(x)dx} + e^{\int\limits_{z_n}^{z} a(x)dx} \int\limits_{z_n}^{z} b(y)\, e^{-\int\limits_{z_n}^{y} a(x)dx} dy \\ &= w_n\, e^{\kappa_n(z-z_n)} \frac{(a^{n-1}+\kappa_n)-(a^{n-1}-\kappa_n)e^{-2\kappa_n(z-z_{n-1})}}{(a^{n-1}+\kappa_n)-(a^{n-1}-\kappa_n)e^{-2\kappa_n h_n}} \\ &\quad - b^{n-1} \frac{e^{-\kappa_n(z-z_{n-1})}-e^{\kappa_n(z-z_n)}e^{-\kappa_n h_n}}{(a^{n-1}+\kappa_n)-(a^{n-1}-\kappa_n)e^{-2\kappa_n h_n}}. \end{aligned} \tag{24}$$

Using the gluing conditions (5), we obtain that w_n can be determined recurrently:

$$w_{n-1} = \frac{2\kappa_n}{(a^{n-1}+\kappa_n)-(a^{n-1}-\kappa_n)e^{-2\kappa_n h_n}} \left(w_n e^{-\kappa_n h_n} - b^{n-1} \frac{1-e^{-2\kappa_n h_n}}{2\kappa_n} \right), \quad n = \overline{N,1}. \tag{25}$$

Note that in the case of (25), firstly, the recurrent recalculation is carried out "bottom to top", and, secondly, the recurrent recalculation must start from w_N, which is unknown, but the value of w_0 is known (see (11)). This situation determines the algorithm for solving the continuation problem, which will be presented in the next section.

4 Solving the continuation problem

Define the following functional:

$$F\big[w_N\big] = |w(\nu,0,p;w_N) - \phi(\nu,p)|^2 \equiv |w_0 - \psi(\nu,p)|^2. \tag{26}$$

To point out that the value of the function $w(\nu,0,p;w_N)$ depends on the value of w_N, this is indicated in the function argument.

To determine w_N the minimum of the functional (26) shoud be found.

First of all, let's calculate a^n and b^n ($n=\overline{1,N-1}$) (see (21) and (23)).

To start the minimization process, it is necessary to choose the initial approximation $w_N^{[0]}$ (for this purpose one can select $w_N^{[0]} = -\kappa_N$).

Values w_n ($n=\overline{N-1,0}$) shoud be calculated by ussing $w_N = w_N^{[j]}$ into (25). The value of $w_N^{[j+1]}$ is calculated according to the rule depending on which method of minimizing the functional (26) is used (see Section 6).

The minimization process stops at step J when $\Phi\big[w_N^{[J]}\big] \le \delta$, where δ quite small.

As a result, after completing the minimization process at the point $z=z_N$ two values are obtain:

$$w_z|_{z=z_N} = a^N w_N^{[J]} + b^N \quad \text{and} \quad w|_{z=z_N} = w_N^{[J]}, \tag{27}$$

which means that the stated problem of continuation of the electromagnetic field in the frequency domain has been solved numerically.

5 Strongly convexity of the residual functional

The functional (26) is strongly convex. Let's prove it.

Firstly, from the problem statement (8)-(10) it is easy to see that

$$w\big(\nu,0,p;\xi w_{1,N} + \zeta w_{2,N}\big) = \xi w\big(\nu,0,p;w_{1,N}\big) + \zeta w\big(\nu,0,p;w_{2,N}\big),$$

where ξ and ζ are constants.

Secondly, for a complex number modulus it is easy to prove

$$|\rho z_1 + (1-\rho) z_2|^2 = \rho |z_1|^2 + (1-\rho)|z_2|^2 - \rho(1-\rho)|z_1 - z_2|^2,$$

where ρ is any real number from the interval $[0,1]$.

From these two equalities the following equality follows

$$\Phi\big[\rho w_{1,N} + (1-\rho) w_{2,N}\big] = \rho \Phi\big[w_{1,N}\big] + (1-\rho)\Phi\big[w_{2,N}\big] - \rho(1-\rho)|w_{1,0} - w_{2,0}|^2. \tag{28}$$

From (25) it follows

$$(w_{1,0} - w_{2,0}) = \Xi \cdot (w_{1,N} - w_{2,N}), \quad \Xi = \prod_{n=1}^{N} \frac{2\kappa_n e^{-\kappa_n h_n}}{(a^{n-1} + \kappa_n) - (a^{n-1} - \kappa_n) e^{-2\kappa_n h_n}}. \tag{29}$$

Denote $q = |\Omega|^2$. From (28) and (29) follows

$$F\big[\rho w_{1,N} + (1-\rho) w_{2,N}\big] = \rho F\big[w_{1,N}\big] + (1-\rho) F\big[w_{2,N}\big] - \rho(1-\rho) q |w_{1,N} - w_{2,N}|^2. \tag{30}$$

Equality (30) means, that functional $F\big[w_N\big]$ is strongly convex.

The property of strong convexity means that the functional has a unique minimum point, and the minimization sequence converges to it, and there is also an estimate of convergence rate of the chosen minimization method (see, for example, Vasiliev, 1988).

6 Searching the minimum of the residual functional

To find the minimum of the functional $F\big[w_N\big]$, we can use some gradient method, for example, the conjugate gradient method (see, for example, Vasiliev, 1988). In this case, it is necessary to be able to calculate the functional gradient $\nabla F\big[w_N\big]$. Note that $F\big[w_N\big]: \mathbb{C} \to \mathbb{R}$, and, based on the definition of the functional gradient, the increment of the functional should be represented in form: $\delta F\big[w_N\big] = \big\langle \nabla F\big[w_N\big], \delta w_N \big\rangle + o(\|\delta w_N\|)$ (where δw_N is the increment for the value w_N), i.e. for two complex quantities, the scalar product must be defined as: $\langle \cdot, \cdot \rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$.

For two complex numbers w_1 and w_2 we define the scalar product as

$$\langle w_1, w_2 \rangle = \frac{1}{2}\big(w_1 \overline{w}_2 + \overline{w}_1 w_2\big) = \mathrm{Re}\big\{\overline{w}_1 w_2\big\}, \tag{31}$$

here the overbar denotes complex conjugation. If $w_1 = w_2 = w$, then $\langle w, w \rangle = |w|^2$, i.e. $\|w\| = |w|$. The relation (31) satisfies the properties of a scalar product. The proof of this is based on the geometric interpretation of a complex number: if $w_1 = (\xi_1, \chi_1)$ and $w_2 = (\xi_2, \chi_2)$, then $\langle w_1, w_2 \rangle = \xi_1 \chi_2 + \chi_1 \xi_2$, which is the result of the scalar product of two vectors. The above approach for deriving an expression for a functional gradient and constructing a minimization sequence was first proposed and tested on numerical examples in the work (Karchevsky, 2009).

We obtain an expression for the gradient of the functional $F\big[w_N\big]$. Let the value w_N receive an increment of δw_N, then the value w_0 will receive an increment of δw_0. It is easy to see:

$$\begin{aligned}
\delta F\big[w_N\big] &= F\big[w_N + \delta w_N\big] - F\big[w_N\big] \\
&= |w_0 + \delta w_0 - \psi|^2 - |w_0 - \psi|^2 \\
&= (w_0 + \delta w_0 - \psi)(\overline{w}_0 + \overline{\delta w_0} - \overline{\psi}) - (w_0 + \psi)(\overline{w}_0 - \overline{\psi}) \\
&= \mathrm{Re}\big\{2(\overline{w}_0 - \overline{\psi})\delta w_0\big\} + |\delta w_0|^2.
\end{aligned}$$

Taking into account (29), we get

$$\delta F\big[w_N\big] = \mathrm{Re}\big\{2(\overline{w}_0 - \overline{\phi})\Xi\, \delta w_N\big\} + q|\delta w_N|^2,$$

from which follows the expression for the functional gradient

$$\nabla F\left[w_N\right] = 2(w_0 - \phi)\,\Xi.$$

To construct a minimization sequence to find the minimum of the functional $F\left[w_N\right]$, the conjugate gradient method can be used (see, for example, Vasiliev, 1988). Each subsequent value of the minimization sequence is constructed according to the following rule:

$$w_N^{[j+1]} = w_N^{[j]} - \zeta_j p_j, \quad j = 0, 1, 2, ...,$$

here

$$p_0 = \nabla F\left[w_N^{[0]}\right], \quad p_j = \nabla F\left[w_N^{[j]}\right]) + \theta_j p_{j-1}, \quad \theta_j = \frac{\left|\nabla F\left[w_N^{[j]}\right]\right|^2}{\left|\nabla F\left[w_N^{[j-1]}\right]\right|^2}, \quad j = 1, 2, ...,$$

and the method step ζ_j is a solution to the problem of minimizing a function of one variable:

$$\zeta_j = \arg\min_{\zeta > 0} F\left[w_N^{[j]} - \zeta p_j\right].$$

This minimization problem can be solved, for example, by the golden section method (see, for example, Vasiliev, 1988).

7 Data preprocessing: Butterworth bandpass filter and Hilbert transform

Before solving the continuation problem, preliminary data processing is necessary. Our work experience (Iskakov et al., 2017; Uzakkyzy et al., 2017; Tokseit et al., 2020; Mukanova et al., 2020) and practice of working with OKO-2 ground penetrating radar (Iskakov et al., 2023a,b; Kussainova, 2023) allows us to conclude that the following two data preprocessing steps must be done.

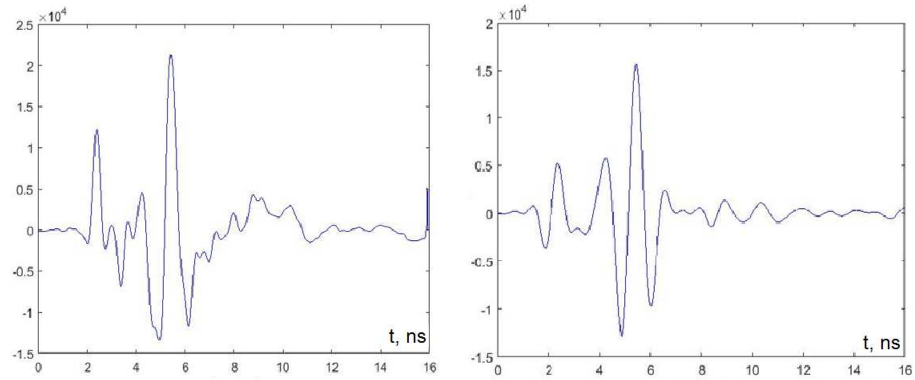

Figure 2: An example of processing one of 3334 radargrams measured on Turan Avenue (Astana) using the OKO-2 georadar with an AB-1000R horn antenna, the main signal frequency is 1 GHz. Radargram (a) – before smoothing and (b) – after smoothing

Step 1: Cleaning data from noise and third-party interference. Initially we have data $\psi(r, t)$, where r takes values from a discrete set of points $\{r_j\}_{j=1}^K$. We smooth each radargram $\psi(r_j, t)$ using a Butterworth bandpass filter (Baskakov, 1988, Bendat & Piersol, 2010, Rapoport, 1993). Fig. 2 shows an example of a radargram before and after smoothing.

Step 2: Refinement of the known part of the environmental model. Application of the Hilbert transform (Baskakov, 1988; Bendat & Piersol, 2010; Rapoport, 1993) - determining the maxima of the image module allows you to clarify the location of the boundaries of the horizontally

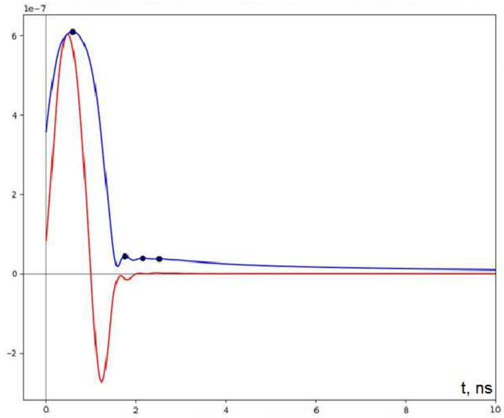

Figure 3: Smoothed radargram (red line) and the modulus of its Hilbert transform image (blue line). Black dots mark places on the radargram corresponding to the boundaries of the layers

layered medium where the signal was reflected, and it also allows you to clarify the location of local inclusions in the medium, parameters which needs to be determined. Fig. 3 shows the smoothed radargram and the Hilbert transform image module. Black dots mark places on the radargram corresponding to the boundaries of the layers.

8 Conclusion

In this work, analytical expressions are obtained for solving the problem of continuation of the electromagnetic field in the frequency domain for a horizontally layered medium. The algorithm for the numerical solution of the problem is based on layer-by-layer recalculation. The resulting analytical expressions are presented in a form that allows calculations and layer-by-layer recalculation without the accumulation of rounding errors. The solution to the continuation problem is obtained as a solution to the cost functional minimization problem. The strong convexity of this cost functional is proven, which implies the existence of a unique solution to the problem posed. Before solving the continuation problem, it is proposed to apply the Butterworth filter to smooth the radargrams and the Hilbert transform technique to clarify the coordinates of the discontinuity points of the medium.

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